Supplementary Material: Implicit Formulation for SPH-based Viscous Fluids

1 Derivation of Implicit Formulation for the Full Form

We show the derivation of our implicit formulation for the full form of viscosity. From the Navier-Stokes equations, we extract the following equations for viscosity:

$$\mathbf{u}_i = \mathbf{u}_i^* + \frac{\Delta t}{\rho_i} \nabla \cdot \mathbf{s}_i, \tag{1}$$

$$\mathbf{s}_i = \mu_i \left(\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T \right), \tag{2}$$

where $\mathbf{u}_i = [u_i, v_i, w_i]^T$ denotes second intermidiate velocity of particle i, $\mathbf{u}_i^*$ first intermediate velocity, Δt time step, ρ_i density, $\mathbf{s}_i$ intermediate viscous stress tensor, and μ_i dynamic viscosity. Using an SPH formulation for divergence of tensor, we obtain

$$\nabla \cdot \mathbf{s}_i = m \rho_i \sum_j \left(\frac{\mathbf{s}_i}{\rho_i^2} + \frac{\mathbf{s}_j}{\rho_j^2} \right) \nabla W_{ij}, \tag{3}$$

and then, by substituting Eq. (3) into Eq. (1), we obtain

$$\mathbf{u}_i = \mathbf{u}_i^* + m \Delta t \sum_j \left(\frac{\mathbf{s}_i}{\rho_i^2} + \frac{\mathbf{s}_j}{\rho_j^2} \right) \nabla W_{ij}. \tag{4}$$

With Eqs. (4) and (2), we can derive

$$\mathbf{u}_i = \mathbf{u}_i^* + m \Delta t \sum_j \left(\frac{\mu_i}{\rho_i^2} \left(\nabla \mathbf{u}_i + (\nabla \mathbf{u}_i)^T \right) + \frac{\mu_j}{\rho_j^2} \left(\nabla \mathbf{u}_j + (\nabla \mathbf{u}_j)^T \right) \right) \nabla W_{ij}, \tag{5}$$

and we obtain the following equation by arranging the terms in Eq. (5):

$$\mathbf{u}_i + m \Delta t \sum_j \left(\frac{\mu_i}{\rho_i^2} \left(-\nabla \mathbf{u}_i - (\nabla \mathbf{u}_i)^T \right) + \frac{\mu_j}{\rho_j^2} \left(-\nabla \mathbf{u}_j - (\nabla \mathbf{u}_j)^T \right) \right) \nabla W_{ij} = \mathbf{u}_i^*. \tag{6}$$

We can rewrite Eq. (6) as

$$\mathbf{u}_i + \hat{m} \sum_j \left(\hat{\mu}_i \mathbf{Q}_{ij} + \hat{\mu}_j \mathbf{Q}_{jk} \right) \nabla W_{ij} = \mathbf{u}_i^*, \tag{7}$$

$$\hat{m} = m \Delta t,$$

$$\hat{\mu}_i = \frac{\mu_i}{\rho_i^2},$$

$$\mathbf{Q}_{ij} = -\nabla \mathbf{u}_i - (\nabla \mathbf{u}_i)^T.$$

Since we can estimate Jacobian of velocity $\nabla \mathbf{u}_i$ using an SPH formulation as

$$\begin{aligned}
\nabla \mathbf{u}_i &= \sum_j V_j (\mathbf{u}_j - \mathbf{u}_i) \nabla W_{ij}^T \\
&= \sum_j V_j \begin{bmatrix} u_j - u_i \\ v_j - v_i \\ w_j - w_i \end{bmatrix} \begin{bmatrix} \nabla W_{ij,x} & \nabla W_{ij,y} & \nabla W_{ij,z} \end{bmatrix} \\
&= \sum_j V_j \begin{bmatrix} (u_j - u_i)\nabla W_{ij,x} & (u_j - u_i)\nabla W_{ij,y} & (u_j - u_i)\nabla W_{ij,z} \\ (v_j - v_i)\nabla W_{ij,x} & (v_j - v_i)\nabla W_{ij,y} & (v_j - v_i)\nabla W_{ij,z} \\ (w_j - w_i)\nabla W_{ij,x} & (w_j - w_i)\nabla W_{ij,y} & (w_j - w_i)\nabla W_{ij,z} \end{bmatrix} \\
&= \sum_j \begin{bmatrix} u_{ji}a_{ij,x} & u_{ji}a_{ij,y} & u_{ji}a_{ij,z} \\ v_{ji}a_{ij,x} & v_{ji}a_{ij,y} & v_{ji}a_{ij,z} \\ w_{ji}a_{ij,x} & w_{ji}a_{ij,y} & w_{ji}a_{ij,z} \end{bmatrix},
\end{aligned}$$

where V_i denotes volume, $W_{ij} = [W_{ij,x}, W_{ij,y}, W_{ij,z}]^T$ kernel, $u_{ij} = u_i - u_j$, $v_{ij} = v_i - v_j$, $w_{ij} = w_i - w_j$, and $\mathbf{a}_{ij} = [a_{ij,x}, a_{ij,y}, a_{ij,z}]^T = [V_j \nabla W_{ij,x}, V_j \nabla W_{ij,y}, V_j \nabla W_{ij,z}]^T$, and transposed Jacobian of velocity $(\nabla \mathbf{u}_i)^T$ can be written as

$$(\nabla \mathbf{u}_i)^T = \sum_j \begin{bmatrix} u_{ji}a_{ij,x} & v_{ji}a_{ij,x} & w_{ji}a_{ij,x} \\ u_{ji}a_{ij,y} & v_{ji}a_{ij,y} & w_{ji}a_{ij,y} \\ u_{ji}a_{ij,z} & v_{ji}a_{ij,z} & w_{ji}a_{ij,z} \end{bmatrix},$$

we obtain

$$\mathbf{Q}_{ij} = -\nabla \mathbf{u}_i - (\nabla \mathbf{u}_i)^T = \begin{bmatrix} 2\sum_j a_{ij,x}u_{ij} & q_{ij,xy} & q_{ij,xz} \\ q_{ij,xy} & 2\sum_j a_{ij,y}v_{ij} & q_{ij,yz} \\ q_{ij,xz} & q_{ij,yz} & 2\sum_j a_{ij,z}w_{ij} \end{bmatrix}, \tag{8}$$

$$q_{ij,xy} = \sum_j \left(a_{ij,y}u_{ij} + a_{ij,x}v_{ij}\right), q_{ij,xz} = \sum_j \left(a_{ij,z}u_{ij} + a_{ij,x}w_{ij}\right),$$

$$q_{ij,yz} = \sum_j \left(a_{ij,z}v_{ij} + a_{ij,y}w_{ij}\right).$$

By substituting Eq. (8) into Eq. (7), we obtain an implicit formulation for x component of $\mathbf{u}_i$, u_i:

$$\begin{aligned}
u_i + \hat{m} \sum_j \Bigg(& \hat{\mu}_i \Big(2\nabla W_{ij,x} \sum_j a_{ij,x}u_{ij} + \nabla W_{ij,y} \sum_j (a_{ij,y}u_{ij} + a_{ij,x}v_{ij}) + \nabla W_{ij,z} \sum_j (a_{ij,z}u_{ij} + a_{ij,x}w_{ij}) \Big) + \\
& \hat{\mu}_j \Big(2\nabla W_{ij,x} \sum_k a_{jk,x}u_{jk} + \nabla W_{ij,y} \sum_k (a_{jk,y}u_{jk} + a_{jk,x}v_{jk}) + \nabla W_{ij,z} \sum_k (a_{jk,z}u_{jk} + a_{jk,x}w_{jk}) \Big) \Bigg) = u_i^*.
\end{aligned} \tag{9}$$

Next, to extract coefficients, we separate the terms in Eq. (9) as

$$\begin{aligned}
u_i + \hat{m}\sum_j \Bigg(\hat{\mu}_i \Big(2\nabla W_{ij,x}(\sum_j a_{ij,x}u_i - \sum_j a_{ij,x}u_j) + \\
\nabla W_{ij,y}(\sum_j a_{ij,y}u_i - \sum_j a_{ij,y}u_j + \sum_j a_{ij,x}v_i - \sum_j a_{ij,x}v_j) + \\
\nabla W_{ij,z}(\sum_j a_{ij,z}u_i - \sum_j a_{ij,z}u_j + \sum_j a_{ij,x}w_i - \sum_j a_{ij,x}w_j)\Big) + \\
\hat{\mu}_j \Big(2\nabla W_{ij,x}(\sum_k a_{jk,x}u_j - \sum_k a_{jk,x}u_k) + \\
\nabla W_{ij,y}(\sum_k a_{jk,y}u_j - \sum_k a_{jk,y}u_k + \sum_k a_{jk,x}v_j - \sum_k a_{jk,x}v_k) + \\
\nabla W_{ij,z}(\sum_k a_{jk,z}u_j - \sum_k a_{jk,z}u_k + \sum_k a_{jk,x}w_j - \sum_k a_{jk,x}w_k)\Big)\Bigg) = u_i^*.
\end{aligned} \tag{10}$$

Since u_i, v_i, and w_i are constant when we compute $\sum_j a_{ij,x}$, $\sum_j a_{ij,y}$, and $\sum_j a_{ij,z}$, we can rewrite Eq. (10) with $\alpha_{ij,x} = \sum_j a_{ij,x}$, $\alpha_{ij,y} = \sum_j a_{ij,y}$, and $\alpha_{ij,z} = \sum_j a_{ij,z}$ as

$$\begin{aligned}
u_i + \hat{m}\sum_j \Bigg(\hat{\mu}_i \Big(2\nabla W_{ij,x}(\alpha_{ij,x}u_i - \sum_j a_{ij,x}u_j) + \\
\nabla W_{ij,y}(\alpha_{ij,y}u_i - \sum_j a_{ij,y}u_j + \alpha_{ij,x}v_i - \sum_j a_{ij,x}v_j) + \\
\nabla W_{ij,z}(\alpha_{ij,z}u_i - \sum_j a_{ij,z}u_j + \alpha_{ij,x}w_i - \sum_j a_{ij,x}w_j)\Big) + \\
\hat{\mu}_j \Big(2\nabla W_{ij,x}(\alpha_{jk,x}u_j - \sum_k a_{jk,x}u_k) + \\
\nabla W_{ij,y}(\alpha_{jk,y}u_j - \sum_k a_{jk,y}u_k + \alpha_{jk,x}v_j - \sum_k a_{jk,x}v_k) + \\
\nabla W_{ij,z}(\alpha_{jk,z}u_j - \sum_k a_{jk,z}u_k + \alpha_{jk,x}w_j - \sum_k a_{jk,x}w_k)\Big)\Bigg) = u_i^*.
\end{aligned} \tag{11}$$

Using the distributive law for $\nabla W_{ij,x}$, $\nabla W_{ij,y}$, and $\nabla W_{ij,z}$ we further separate the terms in Eq. (11):

$$\begin{aligned}
u_i + \hat{m}\sum_j \Bigg(\hat{\mu}_i \Big(2\nabla W_{ij,x}\alpha_{ij,x}u_i - 2\nabla W_{ij,x}\sum_j a_{ij,x}u_j + \\
\nabla W_{ij,y}\alpha_{ij,y}u_i - \nabla W_{ij,y}\sum_j a_{ij,y}u_j + \nabla W_{ij,y}\alpha_{ij,x}v_i - \nabla W_{ij,y}\sum_j a_{ij,x}v_j + \\
\nabla W_{ij,z}\alpha_{ij,z}u_i - \nabla W_{ij,z}\sum_j a_{ij,z}u_j + \nabla W_{ij,z}\alpha_{ij,x}w_i - \nabla W_{ij,z}\sum_j a_{ij,x}w_j\Big) + \\
\hat{\mu}_j \Big(2\nabla W_{ij,x}\alpha_{jk,x}u_j - 2\nabla W_{ij,x}\sum_k a_{jk,x}u_k) + \\
\nabla W_{ij,y}\alpha_{jk,y}u_j - \nabla W_{ij,y}\sum_k a_{jk,y}u_k + \nabla W_{ij,y}\alpha_{jk,x}v_j - \nabla W_{ij,y}\sum_k a_{jk,x}v_k + \\
\nabla W_{ij,z}\alpha_{jk,z}u_j - \nabla W_{ij,z}\sum_k a_{jk,z}u_k + \nabla W_{ij,z}\alpha_{jk,x}w_j - \nabla W_{ij,z}\sum_k a_{jk,x}w_k\Big)\Bigg) = u_i^*.
\end{aligned} \tag{12}$$

Since μ_i is constant when we scan particle j, we rewrite Eq. (12) as

$$\begin{aligned}
u_i + \hat{m}\hat{\mu}_i \sum_j \Big(2\nabla W_{ij,x}\alpha_{ij,x}u_i - 2\nabla W_{ij,x}\sum_j a_{ij,x}u_j + \\
\nabla W_{ij,y}\alpha_{ij,y}u_i - \nabla W_{ij,y}\sum_j a_{ij,y}u_j + \nabla W_{ij,y}\alpha_{ij,x}v_i - \nabla W_{ij,y}\sum_j a_{ij,x}v_j + \\
\nabla W_{ij,z}\alpha_{ij,z}u_i - \nabla W_{ij,z}\sum_j a_{ij,z}u_j + \nabla W_{ij,z}\alpha_{ij,x}w_i - \nabla W_{ij,z}\sum_j a_{ij,x}w_j \Big) + \\
\hat{m}\sum_j \hat{\mu}_j \Big(2\nabla W_{ij,x}\alpha_{jk,x}u_j - 2\nabla W_{ij,x}\sum_k a_{jk,x}u_k + \\
\nabla W_{ij,y}\alpha_{jk,y}u_j - \nabla W_{ij,y}\sum_k a_{jk,y}u_k + \nabla W_{ij,y}\alpha_{jk,x}v_j - \nabla W_{ij,y}\sum_k a_{jk,x}v_k + \\
\nabla W_{ij,z}\alpha_{jk,z}u_j - \nabla W_{ij,z}\sum_k a_{jk,z}u_k + \nabla W_{ij,z}\alpha_{jk,x}w_j - \nabla W_{ij,z}\sum_k a_{jk,x}w_k \Big) = u_i^*.
\end{aligned} \tag{13}$$

Then, we further decompose Eq. (13), noting that when we scan particle j, α_{ij} is constant, and α_{jk} is not constant:

$$\begin{aligned}
u_i + 2\hat{m}\hat{\mu}_i\alpha_{ij,x}u_i\sum_j \nabla W_{ij,x} - 2\hat{m}\hat{\mu}_i\sum_j \nabla W_{ij,x}\sum_j a_{ij,x}u_j + \\
\hat{m}\hat{\mu}_i\alpha_{ij,y}u_i\sum_j \nabla W_{ij,y} - \hat{m}\hat{\mu}_i\sum_j \nabla W_{ij,y}\sum_j a_{ij,y}u_j + \hat{m}\hat{\mu}_i\alpha_{ij,x}v_i\sum_j \nabla W_{ij,y} - \hat{m}\hat{\mu}_i\sum_j \nabla W_{ij,y}\sum_j a_{ij,x}v_j + \\
\hat{m}\hat{\mu}_i\alpha_{ij,z}u_i\sum_j \nabla W_{ij,z} - \hat{m}\hat{\mu}_i\sum_j \nabla W_{ij,z}\sum_j a_{ij,z}u_j + \hat{m}\hat{\mu}_i\alpha_{ij,x}w_i\sum_j \nabla W_{ij,z} - \hat{m}\hat{\mu}_i\sum_j \nabla W_{ij,z}\sum_j a_{ij,x}w_j + \\
2\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,x}\alpha_{jk,x}u_j - 2\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,x}\sum_k a_{jk,x}u_k + \\
\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,y}u_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\sum_k a_{jk,y}u_k + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,x}v_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\sum_k a_{jk,x}v_k + \\
\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,z}u_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\sum_k a_{jk,z}u_k + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,x}w_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\sum_k a_{jk,x}w_k = u_i^*.
\end{aligned} \tag{14}$$

With $\omega_{ij,x} = \sum_j \nabla W_{ij,x}, \omega_{ij,y} = \sum_j \nabla W_{ij,y}$, and $\omega_{ij,z} = \sum_j \nabla W_{ij,z}$, we can rewrite Eq. (14):

$$\begin{aligned}
u_i + 2\hat{m}\hat{\mu}_i\omega_{ij,x}\alpha_{ij,x}u_i - 2\hat{m}\hat{\mu}_i\omega_{ij,x}\sum_j a_{ij,x}u_j + \\
\hat{m}\hat{\mu}_i\omega_{ij,y}\alpha_{ij,y}u_i - \hat{m}\hat{\mu}_i\omega_{ij,y}\sum_j a_{ij,y}u_j + \hat{m}\hat{\mu}_i\omega_{ij,y}\alpha_{ij,x}v_i - \hat{m}\hat{\mu}_i\omega_{ij,y}\sum_j a_{ij,x}v_j + \\
\hat{m}\hat{\mu}_i\omega_{ij,z}\alpha_{ij,z}u_i - \hat{m}\hat{\mu}_i\omega_{ij,z}\sum_j a_{ij,z}u_j + \hat{m}\hat{\mu}_i\omega_{ij,z}\alpha_{ij,x}w_i - \hat{m}\hat{\mu}_i\omega_{ij,z}\sum_j a_{ij,x}w_j + \\
2\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,x}\alpha_{jk,x}u_j - 2\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,x}\sum_k a_{jk,x}u_k + \\
\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,y}u_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\sum_k a_{jk,y}u_k + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,x}v_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\sum_k a_{jk,x}v_k + \\
\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,z}u_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\sum_k a_{jk,z}u_k + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,x}w_j - \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\sum_k a_{jk,x}w_k = u_i^*.
\end{aligned} \tag{15}$$

By grouping the terms in Eq. (15) with respect to $u_i, v_i, w_i, u_j, v_j, w_j, u_k, v_k$, and w_k, we obtain

$$\begin{aligned}
(1 + \hat{m}\hat{\mu}_i(2\omega_{ij,x}\alpha_{ij,x} + \omega_{ij,y}\alpha_{ij,y} + \omega_{ij,z}\alpha_{ij,z}))u_i &+ \\
\hat{m}\hat{\mu}_i\omega_{ij,y}\alpha_{ij,x}v_i &+ \\
\hat{m}\hat{\mu}_i\omega_{ij,z}\alpha_{ij,x}w_i &+ \\
\hat{m}\hat{\mu}_i(-2\omega_{ij,x}\sum_j a_{ij,x}u_j - \omega_{ij,y}\sum_j a_{ij,y}u_j - \omega_{ij,z}\sum_j a_{ij,z}u_j) &+ \\
\hat{m}\sum_j \hat{\mu}_j(2\nabla W_{ij,x}\alpha_{jk,x} + \nabla W_{ij,y}\alpha_{jk,y}\nabla W_{ij,z}\alpha_{jk,z})u_j &+ \\
\hat{m}\hat{\mu}_i(-\omega_{ij,y}\sum_j a_{ij,x}v_j) + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,x}v_j &+ \\
\hat{m}\hat{\mu}_i(-\omega_{ij,z}\sum_j a_{ij,x}w_j) + \hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,x}w_j &+ \\
\hat{m}\sum_j \hat{\mu}_j(-2\nabla W_{ij,x}\sum_k a_{jk,x}u_k - \nabla W_{ij,y}\sum_k a_{jk,y}u_k - \nabla W_{ij,z}\sum_k a_{jk,z}u_k) &+ \\
-\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}\sum_k a_{jk,x}v_k &+ \\
-\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}\sum_k a_{jk,x}w_k &= u_i^*.
\end{aligned} \tag{16}$$

Then, we further convert Eq. (16) into the following equation with coefficients $c_{u_iu_i}, c_{v_iu_i}, c_{w_iu_i}, c_{u_ju_i}, c_{v_ju_i}, c_{w_ju_i}, c_{u_ku_i}, c_{v_ku_i}$, and $c_{w_ku_i}$:

$$\begin{bmatrix} c_{u_iu_i} \\ c_{v_iu_i} \\ c_{w_iu_i} \end{bmatrix}^T \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum_j \begin{bmatrix} c_{u_ju_i} \\ c_{v_ju_i} \\ c_{w_ju_i} \end{bmatrix}^T \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} + \sum_k \begin{bmatrix} c_{u_ku_i} \\ c_{v_ku_i} \\ c_{w_ku_i} \end{bmatrix}^T \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} = u_i^*, \tag{17}$$

$$\begin{aligned}
c_{u_iu_i} &= 1 + \hat{m}\hat{\mu}_i\left(2\omega_{ij,x}\alpha_{ij,x} + \omega_{ij,y}\alpha_{ij,y} + \omega_{ij,z}\alpha_{ij,z}\right), \\
c_{v_iu_i} &= \hat{m}\hat{\mu}_i\omega_{ij,y}\alpha_{ij,x}, \\
c_{w_iu_i} &= \hat{m}\hat{\mu}_i\omega_{ij,z}\alpha_{ij,x},
\end{aligned}$$

$$\begin{aligned}
c_{u_ju_i} = \hat{m}\big(&-\hat{\mu}_i(2a_{ij,x}\omega_{ij,x} + a_{ij,y}\omega_{ij,y} + a_{ij,z}\omega_{ij,z}) + \\
&\hat{\mu}_j(2\nabla W_{ij,x}\alpha_{jk,x} + \nabla W_{ij,y}\alpha_{jk,y} + \nabla W_{ij,z}\alpha_{jk,z})\big), \\
c_{v_ju_i} &= \hat{m}\left(-\hat{\mu}_i a_{ij,x}\omega_{ij,y} + \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,x}\right), \\
c_{w_ju_i} &= \hat{m}\left(-\hat{\mu}_i a_{ij,x}\omega_{ij,z} + \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,x}\right),
\end{aligned}$$

$$\begin{aligned}
c_{u_ku_i} &= -\hat{m}\sum_j \hat{\mu}_j(2\nabla W_{ij,x}a_{jk,x} + \nabla W_{ij,y}a_{jk,y} + \nabla W_{ij,z}a_{jk,z}), \\
c_{v_ku_i} &= -\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,y}a_{jk,x}, \\
c_{w_ku_i} &= -\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}a_{jk,x}.
\end{aligned}$$

We use $c_{u_iu_i}$ to denote a coefficient of u_i to u_i, and $c_{v_iu_i}$ a coefficient of v_i to u_i, and similarly define other coefficients. Note that we convert our implicit formulation into Eq. (17) to extract coefficients, prioritizing clarity and simplicity, and thus Eq. (17) can be optimized by reducing the number of arithmetic operations with computed values in the extraction loops.

Similarly, for v_i, we compute coefficients $c_{u_iv_i}, c_{v_iv_i}, c_{w_iv_i}, c_{u_jv_i}, c_{v_jv_i}, c_{w_jv_i}, c_{u_kv_i}, c_{v_kv_i}$, and $c_{w_kv_i}$:

$$\begin{bmatrix} c_{u_iv_i} \\ c_{v_iv_i} \\ c_{w_iv_i} \end{bmatrix}^T \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum_j \begin{bmatrix} c_{u_jv_i} \\ c_{v_jv_i} \\ c_{w_jv_i} \end{bmatrix}^T \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} + \sum_k \begin{bmatrix} c_{u_kv_i} \\ c_{v_kv_i} \\ c_{w_kv_i} \end{bmatrix}^T \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} = v_i^*, \tag{18}$$

$$c_{u_i,v_i} = \hat{m}\hat{\mu}_i\omega_{ij,x}\alpha_{ij,y}, \tag{19}$$

$$c_{v_i,v_i} = 1 + \hat{m}\hat{\mu}_i\left(\omega_{ij,x}\alpha_{ij,x} + 2\omega_{ij,y}\alpha_{ij,y} + \omega_{ij,z}\alpha_{ij,z}\right), \tag{20}$$

$$c_{z_i,v_i} = \hat{m}\hat{\mu}_i\omega_{ij,z}\alpha_{ij,y}, \tag{21}$$

$$c_{u_jv_i} = \hat{m}\left(-\hat{\mu}_i a_{ij,y}\omega_{ij,x} + \hat{\mu}_j\nabla W_{ij,x}\alpha_{jk,y}\right), \tag{22}$$

$$c_{v_jv_i} = \hat{m}\Big(-\hat{\mu}_i(a_{ij,x}\omega_{ij,x} + 2a_{ij,y}\omega_{ij,y} + a_{ij,z}\omega_{ij,z}) + \hat{\mu}_j(\nabla W_{ij,x}\alpha_{jk,x} + 2\nabla W_{ij,y}\alpha_{jk,y} + \nabla W_{ij,z}\alpha_{jk,z})\Big), \tag{23}$$

$$c_{w_jv_i} = \hat{m}\left(-\hat{\mu}_i a_{ij,y}\omega_{ij,z} + \hat{\mu}_j\nabla W_{ij,z}\alpha_{jk,y}\right), \tag{24}$$

$$c_{u_kv_i} = -\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,x}a_{jk,y}, \tag{25}$$

$$c_{v_kv_i} = -\hat{m}\sum_j \hat{\mu}_j(\nabla W_{ij,x}a_{jk,x} + 2\nabla W_{ij,y}a_{jk,y} + \nabla W_{ij,z}a_{jk,z}), \tag{26}$$

$$c_{w_kv_i} = -\hat{m}\sum_j \hat{\mu}_j\nabla W_{ij,z}a_{jk,y}. \tag{27}$$

Similarly, for w_i, we compute coefficients $c_{u_iw_i}, c_{v_iw_i}, c_{w_iw_i}, c_{u_jw_i}, c_{v_jw_i}, c_{w_jw_i}, c_{u_kw_i}, c_{v_kw_i}$, and $c_{w_kw_i}$:

$$\begin{bmatrix} c_{u_iw_i} \\ c_{v_iw_i} \\ c_{w_iw_i} \end{bmatrix}^T \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum_j \begin{bmatrix} c_{u_jw_i} \\ c_{v_jw_i} \\ c_{w_jw_i} \end{bmatrix}^T \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} + \sum_k \begin{bmatrix} c_{u_kw_i} \\ c_{v_kw_i} \\ c_{w_kw_i} \end{bmatrix}^T \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix} = w_i^*, \tag{28}$$

$$c_{u_iw_i} = \hat{m}\hat{\mu}_i\omega_{ij,x}\alpha_{ij,z}, \tag{29}$$

$$c_{v_iw_i} = \hat{m}\hat{\mu}_i\omega_{ij,y}\alpha_{ij,z}, \tag{30}$$

$$c_{w_iw_i} = 1 + \hat{m}\hat{\mu}_i\left(\omega_{ij,x}\alpha_{ij,x} + \omega_{ij,y}\alpha_{ij,y} + 2\omega_{ij,z}\alpha_{ij,z}\right), \tag{31}$$

$$c_{u_jw_i} = \hat{m}\left(-\hat{\mu}_i a_{ij,z}\omega_{ij,x} + \hat{\mu}_j\nabla W_{ij,x}\alpha_{jk,z}\right), \tag{32}$$

$$c_{v_jw_i} = \hat{m}\left(-\hat{\mu}_i a_{ij,z}\omega_{ij,y} + \hat{\mu}_j\nabla W_{ij,y}\alpha_{jk,z}\right), \tag{33}$$

$$c_{w_jw_i} = \hat{m}\Big(-\hat{\mu}_i(a_{ij,x}\omega_{ij,x} + a_{ij,y}\omega_{ij,y} + 2a_{ij,z}\omega_{ij,z}) + \hat{\mu}_j(\nabla W_{ij,x}\alpha_{jk,x} + \nabla W_{ij,y}\alpha_{jk,y} + 2\nabla W_{ij,z}\alpha_{jk,z})\Big), \tag{34}$$

$$c_{u_k w_i} = -\hat{m} \sum_j \hat{\mu}_j \nabla W_{ij,x} a_{jk,z}, \tag{35}$$

$$c_{v_k w_i} = -\hat{m} \sum_j \hat{\mu}_j \nabla W_{ij,y} a_{jk,z}, \tag{36}$$

$$c_{w_k w_i} = -\hat{m} \sum_j \hat{\mu}_j (\nabla W_{ij,x} a_{jk,x} + \nabla W_{ij,y} a_{jk,y} + 2\nabla W_{ij,z} a_{jk,z}). \tag{37}$$