Good-Fit Box for 3D Triangles in $O(n \log n)$ Time

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We are given a collection of triangles, and we wish to compute a good-fit box for them in $O(n \log n)$ time. Our main problem is computing the orientation for the box, after which computing the position and dimensions is straightforward.

First compute the convex hull of the vertices of the triangles. This takes $O(n \log n)$ time. The convex hull is generically composed of triangles – if it contains higher order polygons, triangularize them. Now let the i'th triangle of the convex hull have the vertices $\mathbf{p}^i$, $\mathbf{q}^i$, and $\mathbf{r}^i$. Let the number of triangles in the convex hull be n.

We will let the area of the i'th triangle to be denoted as A^i,

$$A^i = \frac{1}{2}|(\mathbf{p}^i - \mathbf{q}^i) \times (\mathbf{p}^i - \mathbf{r}^i)|$$

The quantity A_H is the surface area of the convex hull, which is just the sum of all areas of the triangular facets of the convex hull,

$$A_H = \sum_i A^i$$

The centroid of the i'th triangle we will denote by the vector $\mathbf{m}^i$,

$$\mathbf{m}^i = (\mathbf{p}^i + \mathbf{q}^i + \mathbf{r}^i)/3$$

The centroid of the entire convex hull is a weighted mean of the triangle centroids, with the triangle area being the weights. The convex hull mean is,

$$\mathbf{m}_H = \frac{\sum_i A^i \mathbf{m}^i}{\sum_i A^i} = \frac{\sum_i A^i \mathbf{m}^i}{A_H}$$

Vectors are given in boldface. The superscripts refer to which triangle of the convex hull is associated with a given quantity. The subscripts refer to which coordinate is being taken (i.e. x, y, or z).

First compute the matrix,

$$C'_{ij} = \int_H f_{ij}\, d\sigma = \sum_{k=1}^{n} \frac{A^k}{12}[9\mathbf{m}_i^k \mathbf{m}_j^k + \mathbf{p}_i^k \mathbf{p}_j^k + \mathbf{q}_i^k \mathbf{q}_j^k + \mathbf{r}_i^k \mathbf{r}_j^k] - \mathbf{m}_i \mathbf{m}_j A_H$$

Next, compute the eigenvectors of C'. The eigenvectors will be orthogonal,and will form the axes of our good fit box.

This matrix C' is actually a scaled version of the true covariance matrix of the convex hull – but since we only want the eigenvectors of the covariance matrix, we can use the scaled matrix (eigenvectors are invariant under uniform scaling).

Once the axis vectors are known, compute the maximum and minimum extents of the original triangle set along each axis. Take the midpoints of these extents to find the center of the box, and take the difference of these extents to get the dimensions of the box (if you want midpoint/size representation).

The computation of C' from the convex hull takes at most linear time, and getting the eigenvectors from C' is a constant time operation. Computing the box placement and size once the axis vectors are known is a linear time operation. So, the convex hull operation dominates the procedure, which is $O(n \log n)$ time.

The procedure can be made faster by skipping the convex hull step, and just using the triangles in the original triangle set, resulting in a linear time procedure – but beware, this yields boxes which can have very bad fit (arbitrarily bad, in fact), resulting from bad distributions or degenerate triangles.